Eigenwave spectrum of surface acoustic waves on a rough self-affine fractal surface

Palasantzas, George

Published in:
Physical Review B

DOI:
10.1103/PhysRevB.50.18670

Eigenwave spectrum of surface acoustic waves on a rough self-affine fractal surface

George Palasantzas
Physics Department, Northeastern University, Boston, Massachusetts 02115
(Received 29 June 1994)

The propagation of a sound wave along a statistically rough solid-vacuum interface is investigated for the case of self-affine fractals. The wave-number relation $\omega=\omega(k)$ is examined for the transverse polarized surface wave. The range of existence of this wave is analyzed as a function of the degree of surface irregularity or fractal dimension.

INTRODUCTION

The propagation of sound in the presence of a nonuniform boundary surface constitutes a major problem in acoustics. One important aspect of the problem is the calculation of the eigenwave spectrum for acoustic waves near a rough surface.[1] In the case of a wavy (periodically) rough surface, the existence of a slow acoustic mode that propagates along the surface has been shown in earlier studies of the problem.[2] In fact, when sound propagates along a rough solid-vacuum interface, the surface supports two mutually compensating acoustic pressure components. As a result the formation of a transverse polarized surface wave takes place. On a smooth surface, shear sound has only one normal pressure component, whereas in a rough surface the additional component has its origin in scattering of volume sound modes by surface irregularities.[3]

Up to now the eigenwave spectrum has been examined for the case of random rough boundaries which are characterized by a Gaussian height-height correlation function[3]

$$C(X)=\sigma^2 e^{-X^2/\xi^2}, \quad C(K)=\frac{\xi}{2\sqrt{\pi}}e^{-k^2\xi^2/4} . \tag{1}$$

The parameters σ and ξ are, respectively, the saturated value of root-mean-square roughness at large length scales, and the in-plane correlation length. In fact, the relevant quantity during the calculations of the eigenwave spectrum is the Fourier transform of $C(X)$ which is denoted by $C(k)$. The authors in Ref. 3 concluded that the range of existence of the surface sound wave is confined in the regime $\beta=2\pi(\xi/\lambda)<1$ where λ is the sound wavelength. For $\beta>1$ the surface roughness brings some additional damping of the volume surface modes. This result is in fact strongly related to the particular case of the height-height correlation function through Eq. (1) which corresponds to a smooth hill-valley surface morphology.

However, a wide variety of surfaces and interfaces occurring in nature is well represented by a kind of roughness associated with self-affine fractal scaling, as defined by Mandelbrodt in terms of fractional Brownian motion.[4] Examples include the nanometer scale topology of vapor-deposited films,[5] the spatial fluctuations of liquid-gas interfaces,[6] and the kilometer scale structures of mountain terrain.[4] Physical processes which produce such surfaces include fracture, erosion, and molecular-beam epitaxy, as well as fluid invasion of porous media.[7] See also Ref. 8 for further review on the subject of self-affine scaling. For these types of surfaces apart from the parameters σ and ξ, a third one enters the scenario which characterizes the degree of surface irregularity (jaggedness). The latter is described by a roughness exponent H, and is associated with a local fractal dimension $D=2(3)-H$ for one- (two-) dimensional surfaces.[4,8] In fact, for $H\sim 1$ a smooth hill-valley structure is obtained, whereas for $H\sim 0$ the surface structure is very jagged [see Fig. 2]. Therefore, an investigation of the effect of the roughness exponent H on the eigenwave spectrum is in order, and more specifically on the regime of existence of the surface sound wave which for the particular model of Gaussian roughness ($H=1$) is given by $\beta<1$.

ROUGHNESS MODEL

Let us denote by $z=h(x)$ the function that describes the interface between vacuum and solid. The function $h(x)$ is assumed to be a random single valued as a function of the in-plane position x. The surface is assumed to be statistically stationary up to second order (translationally invariant) such that

$$\langle h(x)\rangle=0, \quad \langle [h(x)-h(x')]^2\rangle=g(X) , \tag{2}$$

where $g(X)=2\sigma^2-2C(X)$. The symbol $\langle\ \rangle$ means a statistical ensemble of rough surfaces, and $X=|x-x'|$. For the case of self-affine fractals it is expected that

$$\begin{aligned} g(X)&\sim X^{2H}, \quad X\ll\xi \\ &=2\sigma^2, \quad X\gg\xi , \end{aligned} \tag{3}$$

where $\sigma=\langle h(x)^2\rangle^{1/2}$, and $0<H<1$.[8] The power-law behavior of $\sim X^{2H}$ at $X\ll\xi$ in Eq. (3) implies a power-law behavior in k space for the Fourier transform of $C(X)$ of the form $C(k)\propto k^{-1-2H}$ in the regime of wave vectors $k\cdot\xi\gg 1$. In a similar manner to that used for the two-dimensional case,[9,10] we shall start our consideration from the Fourier space for the height-height correlation function (see also Ref. 10 for other correlation forms).

We define the Fourier transform of $h(x)$, and $C(X)$, respectively, by

$$h(k)=\frac{1}{2\pi}\int h(x)e^{-ikx}dx\ ,$$
$$C(X)=\frac{1}{L}\int \langle h(x)h(x+X)\rangle dx\ , \tag{4}$$

where L is the in-plane macroscopic length over which the correlations are considered ($L \gg \xi$). Then from Eqs. (4) we obtain[11]

$$\langle |h(k)|^2\rangle=\frac{L}{(2\pi)^2}C(k)\ ,$$
$$C(k)=\int C(X)e^{-ikX}dX\ . \tag{5}$$

CORRELATION MODEL

Following previous work for two-dimensional surfaces,[9] we shall examine the possibility of a class of correlation functions in k space of the form

$$\langle |h(k)|^2\rangle=\frac{L}{(2\pi)^2}\frac{\sigma^2\xi}{(1+a|k|\xi)^{1+2H}} \tag{6}$$

which at large wave vectors k [or small length scales, see Eq. (3)] display power-law behavior $\sim k^{1+2H}$. The parameter a in Eq. (6) is determined from the normalization condition $[(2\pi)^2/L]\int_{-k_c}^{+k_c}\langle |h(k)|^2\rangle dk=\sigma^2$. The parameter k_c is given by $k_c=\pi/a_0$ with a_0 the atomic spacing, since for length scales lower than a_0 any notion of continuum treatment becomes meaningless. After substitution in the normalization relation, we obtain

$$a=\begin{cases}\dfrac{1}{H}[1-(1+ak_c\xi)^{-2H}], & H>0\\ 2\ln(1+ak_c\xi), & H\to 0\end{cases} \tag{7}$$

where the limit $H\to 0$ is obtained through the formula $\lim_{H\to 0}(1/H)[x^H-1]=\ln(x)$. For $H>0$ and $\xi \gg a_0$ we obtain from Eq. (7), $a\approx 1/H$. The associated form of the correlation function is given by

$$C(X)=2\sigma^2\xi\int_0^{k_c}\frac{\cos(kX)}{(1+ak\xi)^{1+2H}}dk\ . \tag{8}$$

EIGENWAVE SPECTRUM

We consider a sound wave with nonzero polarization along the y axis which will be denoted by $u(x,z,t)$. The elasticity theory equation for this case has the form

$$\partial_t^2 u(x,z,t)=c^2\nabla^2 u(x,z,t)\ , \tag{9}$$

where c is the shear sound velocity in the solid. Since the problem is uniform along the y axis, all the y derivatives will be zero. Let us denote by T_{ij} the stress tensor, and the unit vector normal to the surface by

$$n=g(-[\partial_x h(x)],0,1),\quad g=\{1+[\partial x_h(x)]^2\}^{-1/2}\ .$$

The acoustic field $u(x,z,t)$ at the boundary $z=h(x)$ should obey the condition

$$T_{i,j}n_j=0 \quad \text{for } z=h(x)\ . \tag{10}$$

If we consider the case of surfaces of small slope or $\partial_x h(x)\ll 1$, and of "low" height or $\sigma \ll \lambda$, the boundary condition given by Eq. (10) can be expanded about the plane $z=0$ according to relation $u(x,z,t)\approx u(o,x,t)+\partial_z u|_{z=0}h(x)$. Taking into account the fact that for an isotropic solid and for $i\neq j$,[12] $T_{i,j}=\mu(\partial_j u_i+\partial_i u_j)$ $(i,j=x,y,z)$, we obtain finally

$$\partial_z u-\partial_x h(x)\partial_x u+h(x)\partial_z^2 u|_{z=0}=0\ . \tag{11}$$

Furthermore, the acoustic field is divided into two

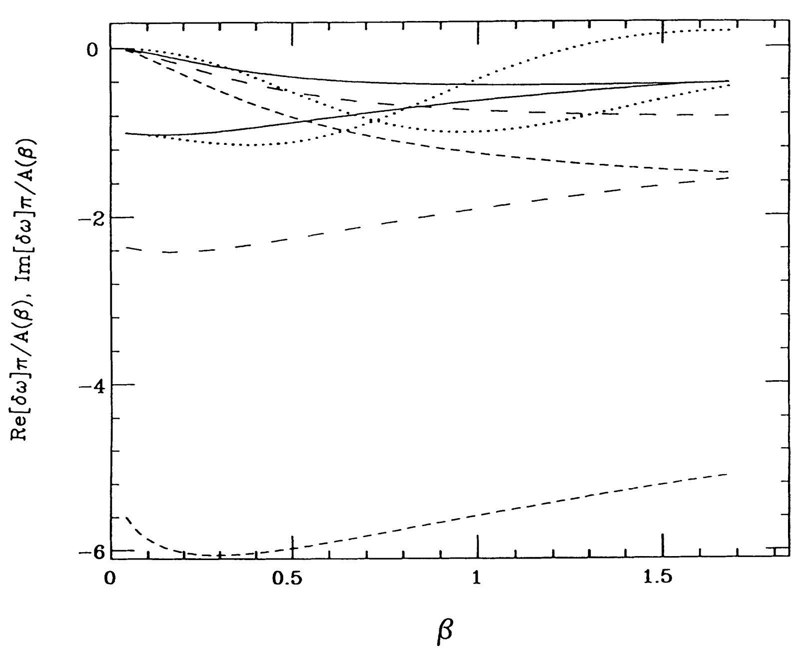

FIG. 1. Schematics of $\mathrm{Re}[\delta\omega]\pi/A(\beta)$ and $\mathrm{Im}[\delta\omega]\pi/A(\beta)$ as a function of β for various values of H. Dots, Gaussian correlation; solid line, $H=0.9$; dashed line, $H=0.7$; and medium dashed line, $H=0.6$. For each set of curves of the same symbol, the upper represents the $\mathrm{Im}[\delta\omega]$ and the lower the Re $[\delta\omega]$.

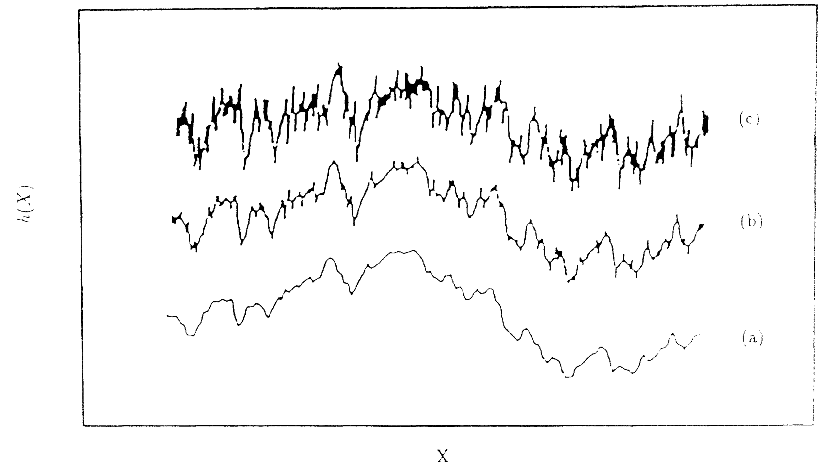

FIG. 2. Schematics of the surface morphology for (a) $H=0.8$, (b) $H=0.5$, and (c) $H=0.2$. Observe the increment in irregularity as H decreases.

parts, namely, $u=\langle u\rangle+u_f$: $\langle u\rangle$ its average part, and u_f its fluctuating part.[3] As is shown by the authors in Ref. 3, the desired equation of the eigenwave spectrum $\omega=\omega(k)$ is given finally by

$$\omega_r(k_x)=\omega_s(k_x)+\delta\omega(k_x)\ , \tag{12}$$

where $\omega_s(k_x)=ck_x$, and $\delta\omega(k_x)=-8\sigma^4k_x^7cF(k_x)$. We define the functions,

$$f_1(x,k_x)=x^{3/2}(x-1)^{-1/2}C(2k_xx)/\sigma^2\ ,$$

$$f_2(x,k_x)=x^{3/2}(1-x)^{-1/2}C(2k_xx)/\sigma^2\ ,$$

$$f_3(x)=x^{3/2}(x+1)^{-1/2}C(2k_xx)/\sigma^2\ .$$

According to Ref. 3, $F(k_x)$ is given by

$$F(k_x)=\left|\int_1^{+\infty}f_1(x,k_x)dx+i\int_0^1 f_2(x,k_x)dx+\int_0^{+\infty}f_3(x,k_x)dx\right|^2\ , \tag{13}$$

where the first and the third integrals represent scattering of sound waves into surface acoustic modes, while the second one represents scattering into volume modes.

NUMERICAL RESULTS

During our simulations we used the following surface parameters: $\sigma=2\times10^{-4}$ cm, $\xi=1.0\times10^{-3}$ cm, and H in the range $0.5<H<1$. For the parameter a we used the approximation $a\sim1/H$ for the choice of ξ and H. In that way, we avoided problems related to finite range of integration (upper bound in k space k_c) for the integrals in Eq. (13). The parameters used for the sound wave were $c=3\times10^5$ cm/s, and sound frequencies in the range f: 2–80 MHz. The corresponding sound wavelengths are given by $c=f\lambda$.

A characteristic parameter in Eqs. (12) and (13) is the ratio $\beta=2\pi\xi/\lambda$. In fact, the authors in Ref. 3, for Gaussian correlation found that for $\beta<1$, $|\mathrm{Im}[\delta\omega]|<|\mathrm{Re}[\delta\omega]|$ which determines the range of existence of a surface sound wave. In the opposite case for $\beta>1$, $|\mathrm{Im}[\delta\omega]|>|\mathrm{Re}[\delta\omega]|$, and the surface roughness only brings an additional damping of the volume surface modes. This result, as our simulations show, is in fact limited by the specific form of the Gaussian correlation function Eq. (1). This form can be realized as a limiting case of self-affine fractal correlations for $H=1$ through the functional $C(X)\sim e^{-(X/\xi)^{2H}}$ which has been used extensively in roughness studies as well.[9,10] From Fig. 1 as we can observe, the regime of existence of a surface sound wave is extended to values of the parameter β significantly larger than one, as long as the roughness exponent H acquires values significantly smaller than one.

We display in Fig. 2 the surface height $h(x)$ for various values of H which has been reproduced from Ref. 8. In Fig. 1 there are two curves of the same symbol from which the upper, for example in the regime $\beta\sim0$, corresponds to the plot of $\mathrm{Im}[\delta\omega]\pi/A(\beta)$, whereas the lower to the plot of $\mathrm{Re}[\delta\omega]\pi/A(\beta)$. The function $A(\beta)$ is given by $A(\beta)=2\beta^{-2}\sigma^4k_x^5c$.

In conclusion, the scattering of volume sound modes from surface inhomogeneities is strongly affected by the density of these irregularities in a manner that not only σ and ξ in comparison with λ, but also the associated fractal dimension of the surface ($D=2-H$) can play a significant role for the existence of a surface wave. In fact H gives a measure of the density of the surface height-height fluctuations,[9] where the denser they become (smaller H), the larger is the regime of existence of surface sound waves.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge partial support from NSF Grant No. DMR-9204022, and the hospitality of the VSM-laboratory at Katholieke Universiteit Leuven (Belgium).

[1]L. M. Brekhovskikh, *Waves in Layered Media* (Nauka, Moscow, 1973); F. G. Bass and I. M. Fuks, *Wave Scattering on Randomly Rough Surface* (Nauka, Moscow, 1972), p. 234.
[2]Y. V. Gulyayev and V. P. Plesski, Zh. Tekh. Fiz. **3**, 220 (1977).
[3]A. A. Bulgakov and S. I. Khankina, Solid State Commun. **44**, 55 (1982).
[4]B. B. Mandelbrodt, *The Fractal Geometry of Nature* (Freeman, New York, 1982).
[5]R. Chiarello, V. Panella, J. Krim, and C. Thompson, Phys. Rev. Lett. **67**, 3408 (1991).
[6]J. H. Sikkenk, J. M. J. Van Leeuwen, E. O. Vossnack, and A. F. Bakker, Physica **146A**, 622 (1987).
[7]M. W. Mitchell and D. A. Bonnell, J. Mater. Res. **5**, 2244 (1990); E. A. Eklund *et al.*, Phys. Rev. Lett. **67**, 1759 (1991); J. Krim *et al.*, *ibid.* **70**, 57 (1993); D. A. Kessler *et al.*, *ibid.* **69**, 100 (1992); M. A. Rubio *et al.*, *ibid.* **63**, 1685 (1989); V. K. Horvath *et al.*, J. Phys. A **24**, L25 (1991).
[8]For a review, see F. Family and T. Vicsek, *Dynamics of Fractal Surfaces* (World Scientific, Singapore, 1991).
[9]G. Palasantzas, Phys. Rev. B **48**, 14 472 (1993); **49**, 5785 (1994).
[10]G. Palasantzas and J. Krim, Phys. Rev. B **48**, 2873 (1993).
[11]J. S. Bendat and A. G. Piersol, *Engineering Applications of Correlation Functions, and Spectral Analysis,* 2nd ed. (Wiley, New York, 1986).
[12]E. Dieulesaint and D. Royer, *Elastic Waves in Solids* (Wiley, New York, 1980).